Centered Hardy-Littlewood maximal function on hyperbolic spaces, $p > 1$

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Abstract. In this paper, we present a natural method to prove that the centered Hardy-Littlewood maximal function, M, on real or complex hyperbolic spaces, $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ and $\mathbb{H}^n_c = \mathbb{R}^+ \times \mathrm{H}(2(n-1), 1)$, satisfies the inequality of the type $\|Mf\|_p \le A_p \|f\|_p$ for all $1 < p \le +\infty$ and $f \in L^p$, where $A_p > 0$ is a constant independent of n. This method can easily be adapted to the case of harmonic AN groups and would also be valid for noncompact symmetric spaces.

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1 Introduction

Consider the standard centered Hardy-Littlewood maximal function, $M_{\mathbb{R}^n}$, on $\mathbb{R}^n$ ($n \in \mathbb{N}^*$), i.e.,

$$M_{\mathbb{R}^n} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n, f \in L^1_{loc}(\mathbb{R}^n),$$

where dy is the Lebesgue measure, $|B_{\mathbb{R}^n}(x,r)|$ is the volume of the Euclidean ball with center x and radius $r > 0$.

By the tripling property of the volume, i.e.

$$|B_{\mathbb{R}^n}(x, 3r)| \le 3^n |B_{\mathbb{R}^n}(x,r)|, \quad \forall x \in \mathbb{R}^n, r > 0,$$

we obtain from the Vitali covering lemma that M is of weak type $(1,1)$ with

$$\|M_{\mathbb{R}^n}\|_{L^1 \longrightarrow L^{1,\infty}} \le 3^n.$$

By the fact that$\|M_{\mathbb{R}^n}\|_{L^\infty \longrightarrow L^\infty} = 1$ and the Marcinkiewicz interpolation theorem, we see

$$\|M_{\mathbb{R}^n}\|_{L^p \longrightarrow L^p} \le 3^{\frac{n}{p}} \frac{p}{p-1}.$$

However, by using the Hopf-Dunford-Schwartz maximal ergodic theorem, Stein and Strömberg proved in [51] that there exists a constant $A > 0$ such that:

$$\|M_{\mathbb{R}^n}\|_{L^1 \longrightarrow L^{1,\infty}} \le A\phi(n), \quad \text{with } \phi(n) = n, \quad \forall n \in \mathbb{N}^*. \tag{1.1}$$

For $1 < p < +\infty$, an estimate of type

$$\|M_{\mathbb{R}^n}\|_{L^p \longrightarrow L^p} \leq C_p, \qquad \text{with } C_p > 0 \text{ (independent of } n), \tag{1.2}$$

can be found in [47], [51] or in [48]. Recall that the proof of (1.2) in [51] (or in [48]) is based on certain transference property of the spherical maximal function on $\mathbb{R}^n$. More precisely, we denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ and $d\sigma$ its standard measure. The spherical maximal function for continuous function f is defined by

$$S_{\mathbb{R}^n} f(x) = \sup_{r>0} \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} |f(x - ry)| \, d\sigma(y), \qquad \forall x \in \mathbb{R}^n;$$

then, for $n \geq n_p$ with $n_p = [\frac{p}{p-1}] + 1$, we have

$$\|S_{\mathbb{R}^n}\|_{L^p \longrightarrow L^p} \leq \|S_{\mathbb{R}^{n_p}}\|_{L^p \longrightarrow L^p}. \tag{1.3}$$

We further remark that $S_{\mathbb{R}^n}$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$, cf. [46] and [52] for $n \geq 3$ and [6] for $n = 2$.

When we replace the usual metric of $\mathbb{R}^n$ by the one induced by the Minkowski functional defined by a symmetric convex bounded open set U; by improving the Vitali covering lemma, Stein and Strömberg proved in [51] that there exists a constant $A > 0$ such that

$$\|M_{\mathbb{R}^n,U}\|_{L^1 \longrightarrow L^{1,\infty}} \leq A(n+1) \ln(n+1), \qquad \forall n \in \mathbb{N}^*. \tag{1.4}$$

Also, the estimate of type (1.2) with $p > \frac{3}{2}$ was obtained in [8], [7] and [11]; it remains valid for $1 < p \leq \frac{3}{2}$ under certain conditions on U, cf. [9] for details.

Recently, by using Doob's maximal inequality, Naor and Tao obtained an estimate of type (1.4) in a vast class of measured metric spaces satisfying the doubling property of the volume. More preceisely, when the measured metric space, (M, ρ, μ), satisfies the "strong n-microdoubling property with constant c", i.e.

$$\mu\Big(B(\xi, (1 + \frac{1}{n})r)\Big) \leq c\mu\Big(B(g, r)\Big), \quad \forall g \in M, r > 0, \xi \in B(g, r),$$

there exists a constant $A = A(c)$, which depends only on $c > 0$ such that (1.4) is valid. See [40] for details.

For the subject of L^p continuity of the spherical maximal function, there are a lot of progress. See for example [20] for the case of real hyperbolic spaces of dimension $n \geq 3$ and [24] for $n = 2$; [43], [41], [38] and [22] in the setting of some class of two step nilpotent Lie groups.

On the other hand, concerning the estimates of type (1.2) and (1.3) in the case of Riemannian manifolds or manifolds equipped with a measure and an (essentially) self-adjoint second order differential operator, few is known. For Heisenberg groups, the estimates were obtained by J. Zienkiewicz in [54] following the method of [51].

For other results concerning the estimates (1.1), (1.2), (1.4), as well as the L^p continuity of the spherical maximal function, see for example [49], [37], [50], [42], [38], [1], [40] and references therein.

This paper is the sequel of the series [30]-[34] whose aim was to understand better the inequalities (1.1) and (1.2).

An estimate of type (1.1) is obtained in the setting of Heisenberg groups, $\mathrm{H}(2n, 1)$, for centered Hardy-Littlewood maximal function defined either by Carnot-Carathéodory distance, or by Korányi norm. The proof is based on a uniform lower estimate of the Poisson kernel (i.e. the integral kernel of the Poisson semi-group, there is no relation with that of [15]), see [30] for detail.

We've also examined in [31] the maximal function M_G associated to the Carnot-Carathéodory distance or to the pseudo-distance associated to the fundamental solution of the Grushin operator,

$$\Delta_G = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + (\sum_{i=1}^n x_i^2)\frac{\partial^2}{\partial u^2}.$$

We obtained the estimate (1.1) for M_G.

As we've already mentioned in [31], the above three results can be roughly explained by an estimate of type

$$\inf_{n\geq 3, r>0, g\neq\xi\in B(g,r)} \phi(n)\frac{n}{r^2}|B(g,r)|(-\Delta)^{-1}(g,\xi) > 0, \quad \text{with } \phi(n) = n, \tag{1.5}$$

in the case of Euclidean spaces and Heisenberg groups, or for Grushin operators. In other words, we believe that there exists a close relation between the estimate of type (1.1) (obviously, the volume of the balls and the dimension play a rôle) and the Green function. In fact, the work [31] is motivated by the estimate (1.5). Also, the results of [51], [30] and [31] can be explained by an estimate of type

$$\inf_{n\geq 3, r>0, g\neq\xi\in B(g,r)} \phi(n)\frac{\sqrt{n}}{r}|B(g,r)|(-\Delta)^{-\frac{1}{2}}(g,\xi) > 0. \tag{1.6}$$

Remark that up to a universal constant, the two terms $\frac{n}{r^2}$ and $\frac{\sqrt{n}}{r}$, which appear in (1.5) and (1.6) respectively, are optimal. One observes also that it is sufficient to take $r = 1$ in the above cases thanks to the dilation structure. See [30] and [31] for detail.

In [33], one continued to use this idea (i.e. the relation between estimate of type (1.1) and the Green function) to obtain estimate of type (1.4) in the case of measured metric spaces of exponential volume growth, like the real hyperbolic spaces $\mathbb{H}^n$; notice that we need to modify the estimates (1.5) and (1.6) due to the special structure of $\mathbb{H}^n$. Following the idea of [31], we obtain in [34] the estimate of type (1.1) for H-type groups, $\mathrm{H}(2n, m)$ (cf. §6 below for notations).

The goal of this paper is to continue the above study and to persuade the reader that there exists a close relationship between the estimates of type (1.1) and (1.2) and the Green function. More precisely, we will use the Green function to obtain the estimate of type (1.2) for real or complex hyperbolic spaces.

Recall that for a measured metric space of exponential volume growth, the tripling property of the volume is no longer valid, and the $L^1 \longrightarrow L^{1,\infty}$ continuity of the centered maximal function, M, is no longer valid in general. For example, for all $n \geq 2$ and all

$1 < p_0 < +\infty$, consider $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ equipped with the hyperbolic metric d (cf. (4.1) below) and with the measure

$$d\mu_{n,p_o}(y,x) = y^{-\frac{p_o}{2p_o-1}(n-1)-1}\, dydx,$$

where dx is the Lebesgue measure on $\mathbb{R}^{n-1}$. One knows that $(\mathbb{R}^+ \times \mathbb{R}^{n-1}, d, d\mu_{n,p_o})$ is of exponential volume growth. In this space, M is bounded on L^p for $p > p_0$ but not for $1 \le p < p_0$, cf. [27] for detail and more examples. However, for noncompact symmetric spaces, Clerc and Stein have showed in [12] that M is bounded on L^p for all $p > 1$, and Strömberg has showed in [53] that M is also of weak type $(1,1)$. The typical noncompact symmetric spaces are real hyperbolic spaces, $\mathbb{H}^n$ $(n \ge 2)$. Also, $\mathbb{H}^n$ are typical examples of harmonic AN groups on which the centered maximal function is of weak type $(1,1)$ and bounded on L^p for all $1 < p \le +\infty$, cf. [3].

Denote by M the centered Hardy-Littlewood maximal function on $\mathbb{H}^n$ or on complex hyperbolic spaces $\mathbb{H}^n_c$, the main result of this paper is the following:

Theorem 1.1 *Let $1 < p < +\infty$. Then there exists a constant $c_p > 0$ such that for all $n \ge 2$, we have*

$$\|Mf\|_p \le c_p \|f\|_p, \quad \forall f \in L^p. \tag{1.7}$$

Remark. 1. Our method can easily be adapted to the case of harmonic AN groups; by modifying the proof of Lemma 5 in [54], we could obtain the estimate (1.7) in the case $\mathbb{R}^+ \times \mathrm{H}(2n, m)$ (at least for fixed m). See §6 below for detail.

2. In [33], an estimate of type (1.4) was established for $\mathbb{H}^n$, the method of [33] is also valid for harmonic AN groups.

3. Our proof probably works in the setting of noncompact symmetric spaces.

1.1 Main idea of the proof of Theorem 1.1

As explained above, the Green function will play a crucial rôle. Also, we will only explain the proof of (1.7) in the case of $\mathbb{H}^n$, as it works for the other cases:

Obviously, it suffices to prove the estimate (1.7) for $1 < p < 2$ and for n large enough (i.e. n greater than a certain $n(p)$). One writes first

$$\begin{aligned} Mf(g) &\le \sup_{0<r<\epsilon} \frac{1}{|B(g,r)|} \int_{B(g,r)} |f(\xi)|\, d\mu(\xi) + \int_{B^c(g,\epsilon)} \frac{|f(\xi)|}{|B(g, d(g,\xi))|}\, d\mu(\xi) \\ &= M_\epsilon f(g) + S_\epsilon f(g), \end{aligned}$$

where $\epsilon > 0$ will be determined later.

In [33], we showed that there exists a constant $C > 1$ such that for all $n \ge 2$, $g \in \mathbb{H}^n$ and $d(g,\xi) > 0$, we have

$$\frac{C^{-1}}{1 + d^2(g,\xi)} \le n^2 \frac{1}{d^2(g,\xi)} |B(g, d(g,\xi))| (-\Delta_{\mathbb{H}^n})^{-1}(g,\xi) \le \frac{C}{1 + d^2(g,\xi)}.$$

We have immediately

$$S_\epsilon f(g) \leq C\frac{1+\epsilon^2}{\epsilon^2}n^2(-\Delta_{\mathbb{H}^n})^{-1}(|f|)(g).$$

Combining the fact that

$$\|(-\Delta_{\mathbb{H}^n})^{-1}\|_{L^p\longrightarrow L^p} \leq c_p n^{-2}, \quad 1<p<+\infty,$$

where $c_p>0$ is independent of n, we can easily treat the part at infinity.

By refining the above idea, one can obtain an estimate for the (micro-)part at infinity as follows:

There exists a constant $c_p>0$ such that for n big enough and for $1>\epsilon>c_p\sqrt{\frac{\ln n}{n}}$, we have for all f and $g\in\mathbb{H}^n$,

$$S_\epsilon f(g) \leq 10^2 Cn^2\Big[-\frac{\rho^2}{p'}-\Delta_{\mathbb{H}^n}\Big]^{-1}(|f|)(g),$$

where $p'^{-1}=1-p^{-1}$ and $\rho^2=(n-1)^2/4$ denotes the spectral gap of $-\Delta_{\mathbb{H}^n}$ on $L^2(\mathbb{H}^n)$.

In order to treat the (micro-)local part, by using the separation of variables as in [28] and [29], we show that for fixed $A>0$, there exists two constants $c(A)>0$ and $n(A)$ such that for $n\geq n(A)$ and $0<\epsilon\leq An^{-\frac{1}{4}}$, we have for all continuous f and all $g=(y,x)$,

$$M_\epsilon f(g) \leq c(A)\sup_{s>0} e^{sL_{n-1}}\Big(M_{\mathbb{R}^{n-1}}f(\cdot,x)\Big)(y),$$

where $e^{sL_{n-1}}$ denotes the heat semigroup for the Sturm-Liouville operator $L_{n-1}=y^2\frac{d^2}{dy^2}-(n-2)y\frac{d}{dy}$ in $L^2(\mathbb{R}^+,y^{-n}dy)$ (the origin of this operator can be found easily in the explicit expression for the Laplacian on $\mathbb{H}^n$, cf.(4.2) below).

To finish the proof of (1.7), we only need to remark that

$$\lim_{n\longrightarrow+\infty}\frac{\sqrt{\frac{\ln n}{n}}}{n^{-\frac{1}{4}}}=0,$$

then use the estimate (1.2), the maximal theorem for symmetric diffusion semigroups as well as that

$$\Big\|\Big[-\frac{\rho^2}{p'}-\Delta_{\mathbb{H}^n}\Big]^{-1}\Big\|_{L^p\longrightarrow L^p} \leq \frac{p'}{\rho^2}.$$

In conclusion, the main idea is to use the Green function, the spectral gap, the separation of variables, and the maximal theorem for symmetric diffusion semigroups. We will also use the spherical maximal function for complex hyperbolic spaces and other AN harmonic groups.

1.2 Organization of the paper

The paper is organized as follows: we recall in Section 2 about the heat kernel for the Sturm-Liouville operator $L_\alpha = y^2\frac{d^2}{dy^2} - (\alpha-1)y\frac{d}{dy}$ $(\alpha > 1)$ on $L^2(\mathbb{R}^+, y^{-\alpha-1}dy)$ and give an important lemma in Section 3 which will be useful for the study of the (micro-)local part. We give the proof of (1.7) in Section 4 for real hyperbolic spaces, in Section 5 for complex hyperbolic spaces. And in the end we will explain briefly how the method of this paper can be adapted to the cases of other harmonic AN groups.

1.3 Notations

In what follows, c, c', A, etc. will stand for universal constants which can take different values from one line to another.

For two functions f and g, we say that $f \sim_1 g$ if there exists a constant $A > 1$ such that $A^{-1}f \le g \le Af$, $f = O(g)$ if there exists a constant $A > 0$ such that $|f| \le Ag$.

2 Recall on the heat kernel of the Sturm-Liouville operator $L_\alpha = y^2\frac{d^2}{dy^2} - (\alpha-1)y\frac{d}{dy}$ $(\alpha > 1)$ on $L^2(\mathbb{R}^+, y^{-\alpha-1}dy)$

We recall in this section the heat kernel for a special class of Sturm-Liouville operators, $L_\alpha = y^2\frac{d^2}{dy^2} - (\alpha-1)y\frac{d}{dy}$ $(\alpha > 1)$. We can refer to [45] and the references therein for basic information. We know that L_α, defined initially on $C_o^\infty((0, +\infty))$, is essentially self-adjoint for the measure $y^{-\alpha-1}dy$. Its heat kernel (i.e. the integral kernel of e^{tL_α} $(t > 0)$) can be written as (cf. for example [39] pp. 211-218):

$$e^{tL_\alpha}(y, v) = \frac{1}{\sqrt{4\pi t}}(yv)^{\frac{\alpha}{2}} e^{-\frac{\alpha^2}{4}t} e^{-\frac{\ln^2 \frac{v}{y}}{4t}}, \quad t > 0, y, v > 0. \tag{2.1}$$

In fact, by the change of variable $y = e^s$, the operator L_α becomes

$$A_\alpha = \frac{d^2}{ds^2} - \alpha\frac{d}{ds}, \quad s \in \mathbb{R}.$$

This is the generator of the Brownian motion with drift $-\alpha$ and we have

$$e^{tA_\alpha}f(s) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\alpha^2}{4}t - \frac{(s-r)^2}{4t} + \frac{\alpha}{2}(s+r)} f(r) e^{-\alpha r}\, dr, \quad \forall s \in \mathbb{R},\ f \text{ convenable}.$$

By replacing s by $\ln y$ (resp. r by $\ln v$) in

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{\alpha^2}{4}t - \frac{(s-r)^2}{4t} + \frac{\alpha}{2}(s+r)},$$

we obtain immediately (2.1).

3 An important lemma

We give in this section a lemma, which plays an important rôle in this paper:

Lemma 3.1 *Let $\beta > 0$. We define*

$$F_\beta(s) = \Big(\frac{s}{\beta}\Big)^2 + \ln\Big(1 - \frac{\sinh^2 s}{\sinh^2 \beta}\Big), \qquad 0 \leq s < \beta.$$

Then, there exists two constants $c > 0$ and $0 < c_o \ll 1$ such that

$$0 < \sup_{0\leq s<\beta} F_\beta(s) \leq c\beta^4, \qquad \forall 0 < \beta \leq c_o. \tag{3.1}$$

Proof. Observe that

$$F_\beta(0) = 0, \qquad \lim_{s\longrightarrow\beta^-} F_\beta(s) = -\infty,$$

and that

$$F'_\beta(s) = 2\frac{s}{\beta^2} - \frac{\sinh(2s)}{\sinh^2\beta - \sinh^2 s}, \quad F'_\beta(0) = 0,$$

$$F''_\beta(s) = \frac{2}{\beta^2} - \frac{(\sinh^2\beta - \sinh^2 s)2\cosh(2s) + \sinh^2(2s)}{(\sinh^2\beta - \sinh^2 s)^2}, \quad F''_\beta(0) > 0.$$

There exists then $0 < s_o = s_o(\beta) < \beta$ such that $F_\beta(s_o) = \sup_{0\leq s<\beta} F_\beta(s) > 0$. Fermat's lemma now implies that

$$0 = F'_\beta(s_o) = 2\frac{s_o}{\beta^2} - \frac{\sinh(2s_o)}{\sinh^2\beta - \sinh^2 s_o}, \quad 0 < s_o < \beta,$$

in other words, we have

$$\frac{\sinh(2s_o)}{2s_o} = \Big(\frac{\sinh\beta}{\beta}\Big)^2 - \frac{\sinh^2(s_o)}{\beta^2}.$$

For $0 < s_o < \beta \ll 1$, by Taylor's formula, we can write

$$1 + \frac{(2s_o)^2}{6} + O(s_o^4) = 1 + \frac{\beta^2}{3} + O(\beta^4) - \frac{s_o^2}{\beta^2}[1 + O(s_o^2)].$$

Dividing this by β^2, we obtain that

$$\frac{1}{3} - \frac{s_o^2}{\beta^4}[1 + O(s_o^2)] - \frac{2s_o^2}{3\beta^2} + O(\beta^2) + O(\frac{s_o^4}{\beta^2}) = 0.$$

We can see easily that $\lim_{\beta\longrightarrow 0^+}\frac{s_o^2}{\beta^4}=\frac{1}{3}$. Moreover, by replacing s_o by $\sqrt{\frac{1}{3}}\beta^2K(\beta)$ in the previous identity, we get immediately

$$s_o=\sqrt{\frac{1}{3}}\beta^2\Big(1+O(\beta^2)\Big). \tag{3.2}$$

As a consequence, we have

$$0<\sup_{0\leq s<\beta}F_\beta(s)=F_\beta(s_o)=\Big(\frac{s_o}{\beta}\Big)^2+\ln\Big(1-\frac{\sinh^2 s_o}{\sinh^2\beta}\Big),$$

by using again Taylor's formula, we get that

$$\begin{aligned}F_\beta(s_o)&=\Big(\frac{s_o}{\beta}\Big)^2-\frac{\sinh^2 s_o}{\sinh^2\beta}+O\Big(\Big[\frac{\sinh^2 s_o}{\sinh^2\beta}\Big]^2\Big)\\&=\Big(\frac{s_o}{\beta}\Big)^2-\Big(\frac{s_o}{\beta}\Big)^2\Big(\frac{\sinh s_o}{s_o}\Big)^2\Big(\frac{\sinh\beta}{\beta}\Big)^{-2}+O(\beta^4)\\&=\Big(\frac{s_o}{\beta}\Big)^2-\Big(\frac{s_o}{\beta}\Big)^2[1+O(\beta^2)]+O(\beta^4)=O(\beta^4).\end{aligned}$$

The proof of (3.1) is thus achieved. ■

4 The case of real hyperbolic spaces

4.1 Recalls on $\mathbb{H}^n$

The real hyperbolic space of dimension $n\geq 2$, $\mathbb{H}^n$, can be considered as the space $\mathbb{R}^+\times\mathbb{R}^{n-1}$ equipped with the Riemannian metric $ds^2=\frac{dy^2+dx^2}{y^2}$. The induced Riemanninan measure can be written as $d\mu(y,x)=y^{-n}dydx$ with dx the Lebesgue measure on $\mathbb{R}^{n-1}$, and the induced Riemannian distance is of the form

$$d((y,x),(v,w))=\operatorname{arc\,cosh}\frac{y^2+v^2+|x-w|^2}{2yv},\qquad\forall(y,x),(v,w)\in\mathbb{R}^+\times\mathbb{R}^{n-1}. \tag{4.1}$$

We have the following expression of Laplacian $\Delta_{\mathbb{H}^n}$:

$$\Delta_{\mathbb{H}^n}=y^2\frac{\partial^2}{\partial y^2}-(n-2)y\frac{\partial}{\partial y}+y^2\Delta_{\mathbb{R}^{n-1}}, \tag{4.2}$$

where $\Delta_{\mathbb{R}^{n-1}}$ is the Laplacian on $\mathbb{R}^{n-1}$. The spectral gap of $-\Delta_{\mathbb{H}^n}$ on $L^2(\mathbb{H}^n)$ is

$$\rho^2=\rho(n)^2=\Big(\frac{n-1}{2}\Big)^2. \tag{4.3}$$

Now we recall the estimates of ball volumes in $\mathbb{H}^n$. Observe that $|B(g,r)|$ does not depend on $g \in \mathbb{H}^n$, we note in the following

$$V(r) = |B(g,r)|, \quad \Psi(r) = (\sinh r)^{n-1} \min\{1, \sinh r\}.$$

The area of the unit sphere and the volume of the unit ball of $\mathbb{R}^n$, ω_{n-1} and Ω_n, are giving respectively by

$$\omega_{n-1} = 2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \qquad \Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{4.4}$$

And we know well that there exists a constant $C_* > 0$, independent of $n \geq 2$, such that (cf. Proposition 2.1 of [33]):

$$C_*^{-1}\Omega_n \Psi(r) \leq V(r) \leq C_* \Omega_n \Psi(r), \qquad \forall r > 0. \tag{4.5}$$

Notice that the heat kernel on $\mathbb{H}^n$, $K_n(t,g,\xi)$, is a function of $(t, d(g,\xi))$, and we define $K_n(t,r)$ $(t > 0,\ r \geq 0)$ as

$$K_n(t,\varsigma) = K_n(t,g,\xi), \qquad \text{with } \varsigma = d(g,\xi).$$

Put

$$K_1(t,r) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t}}.$$

It is well-known (cf. eg. [19]):

$$K_{n+2}(t,r) = e^{-nt}\Big(-\frac{1}{2\pi}\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)K_n(t,r) = e^{-nt}\Big(-\frac{1}{2\pi}\frac{\partial}{\partial \phi}\Big)\Big|_{\phi=\cosh r} K_n(t, \operatorname{arccosh}\phi), \tag{4.6}$$

$$K_n(t,r) = \sqrt{2}e^{\frac{2n-1}{4}t}\int_r^{+\infty} K_{n+1}(t,s)\frac{\sinh s}{\sqrt{\cosh s - \cosh r}}\, ds, \tag{4.7}$$

for all $n \in \mathbb{N}^+$, $t > 0$ and $r \geq 0$. In particular, we have

$$K_2(t,r) = \sqrt{2}(4\pi t)^{-\frac{3}{2}} e^{-\frac{t}{4}} \int_r^{+\infty} \frac{s e^{-\frac{s^2}{4t}}}{\sqrt{\cosh s - \cosh r}}\, ds. \tag{4.8}$$

4.2 The explicit expression for the Green function $(\lambda - \Delta_{\mathbb{H}^n})^{-1}$ $(\lambda > -\rho^2)$

In [36], Matsumoto used a probabilistic method to find an explicit expression of $(\lambda - \Delta_{\mathbb{H}^n})^{-1}$ with $\lambda \geq 0$. More precisely, he obtained in Theorem 3.3 of [36],

$$(\lambda - \Delta_{\mathbb{H}^n})^{-1}(g,\xi) = (2\pi)^{-\frac{n}{2}}(\sinh\varsigma)^{-\frac{n-2}{2}} e^{-\imath(\pi\frac{n-2}{2})} Q^{\frac{n-2}{2}}_{\theta_n(\lambda)}(\cosh\varsigma), \quad \forall g \neq \xi, \tag{4.9}$$

with

$$\theta_n(\lambda) = \sqrt{\lambda+\rho^2} - \frac{1}{2} = \sqrt{\lambda + \frac{(n-1)^2}{4}} - \frac{1}{2}, \qquad \varsigma = d(g,\xi), \tag{4.10}$$

and the Legendre function of second type $Q_\eta^\gamma(\cosh r)$ (with $\eta, \gamma > 0$ and $r \geq 0$) is defined by (cf.[21] p. 155):

$$e^{-\imath(\pi\gamma)} Q_\eta^\gamma(\cosh r) = 2^{-\eta-1} \frac{\Gamma(\eta+\gamma+1)}{\Gamma(\eta+1)} (\sinh r)^{-\gamma} \int_0^\pi (\cosh r + \cos t)^{\gamma-\eta-1} (\sin t)^{2\eta+1}\, dt. \tag{4.11}$$

By analytic extension, the expression (4.9) remains valid for $\lambda > -\rho^2$. As this expression will be important for this paper, we give here

An analytic proof of (4.9)**.**
To simplify the notations, we define

$$G(n,\lambda,r)(n \geq 2, \lambda > -\rho^2, r > 0) \quad \text{by} \quad G(n,\lambda,\varsigma) = (\lambda - \Delta_{\mathbb{H}^n})^{-1}(g,\xi).$$

Consider first the case where $n = 2j+2$ ($j \geq 0$). By (4.6), we have

$$\begin{aligned} K_{2j+2}(t,r) &= \exp\Big\{-\sum_{k=1}^{j}(2k)t\Big\}\Big(-\frac{1}{2\pi}\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^j K_2(t,r) \\ &= e^{-j(j+1)t}(-2\pi)^{-j}\Big(\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^j K_2(t,r). \end{aligned}$$

So,

$$\begin{aligned} G(2j+2,\lambda,r) &= \int_0^{+\infty} e^{-\lambda t} K_{2j+2}(t,r)\, dt \\ &= (-2\pi)^{-j}\Big(\frac{\partial}{\partial \cosh r}\Big)^j \int_0^{+\infty} e^{-[\lambda+j(j+1)]t} K_2(t,r)\, dt. \end{aligned}$$

Then, (4.8) and Fubini's theorem imply that

$$\begin{aligned} &\int_0^{+\infty} e^{-[\lambda+j(j+1)]t} K_2(t,r)\, dt \\ &= \sqrt{2}(4\pi)^{-\frac{3}{2}} \int_r^{+\infty} \frac{s}{\sqrt{\cosh s - \cosh r}} \Big\{ \int_0^{+\infty} e^{-[\lambda+(\frac{2j+1}{2})^2]t} t^{-\frac{3}{2}} e^{-\frac{s^2}{4t}}\, dt \Big\}\, ds. \end{aligned}$$

However, by the change of variable $t = \frac{1}{h}$, we have

$$\begin{aligned} \int_0^{+\infty} e^{-[\lambda+(\frac{2j+1}{2})^2]t} t^{-\frac{3}{2}} e^{-\frac{s^2}{4t}}\, dt &= \int_0^{+\infty} h^{-\frac{1}{2}} e^{-\frac{s^2}{4}h - \frac{\lambda+(\frac{2j+1}{2})^2}{h}}\, dh \\ &= 2\sqrt{\pi}\frac{1}{s} e^{-\sqrt{\lambda+(\frac{2j+1}{2})^2}\, s}, \quad \text{(cf. §3.471 15 of [23] (p. 369)).} \end{aligned}$$

We get then for $n = 2j+2$,

$$\int_0^{+\infty} e^{-[\lambda+j(j+1)]t} K_2(t,r)\,dt = \sqrt{2}2\sqrt{\pi}(4\pi)^{-\frac{3}{2}} \int_r^{+\infty} \frac{e^{-\sqrt{\lambda+(\frac{2j+1}{2})^2}s}}{\sqrt{\cosh s - \cosh r}}\,ds$$
$$= \frac{1}{2\pi} Q_{\theta_n(\lambda)}(\cosh r), \text{(cf. §8.715 2 of [23] (p. 962))}.$$

As a consequence,

$$\begin{aligned}
G(2j+2,\lambda,r) &= \frac{1}{2\pi}(-2\pi)^{-j}\Big(\frac{\partial}{\partial \cosh r}\Big)^j Q_{\theta_n(\lambda)}(\cosh r)\\
&= \frac{1}{2\pi}(-2\pi)^{-j}(\cosh^2 r - 1)^{-\frac{j}{2}} Q^j_{\theta_n(\lambda)}(\cosh r) \text{ (cf.§8.752 4 of [23] p. 968)}\\
&= (2\pi)^{-\frac{n}{2}}(\sinh r)^{-\frac{n-2}{2}} e^{-\imath(\pi\frac{n-2}{2})} Q^{\frac{n-2}{2}}_{\theta_n(\lambda)}(\cosh r).
\end{aligned}$$

Consider now the case where $n = 2j+1$. We observe first by (4.7) that

$$\begin{aligned}
G(2j+1,\lambda,r) &= \int_0^{+\infty} e^{-\lambda t} K_{2j+1}(t,r)\,dt\\
&= \sqrt{2}\int_0^{+\infty} e^{-\lambda t}\Big\{ e^{\frac{2(2j+1)-1}{4}t} \int_r^{+\infty} K_{2j+2}(t,s)\frac{\sinh s}{\sqrt{\cosh s - \cosh r}}\,ds\Big\}\,dt,
\end{aligned}$$

then, by Fubini's theorem, we have

$$G(2j+1,\lambda,r) = \sqrt{2}\int_r^{+\infty} G(2j+2,\lambda - \frac{2(2j+1)-1}{4}, s)\frac{\sinh s}{\sqrt{\cosh s - \cosh r}}\,ds.$$

As

$$\lambda - \frac{2(2j+1)-1}{4} + \Big(\frac{2j+1}{2}\Big)^2 = \lambda + \Big(\frac{(2j+1)-1}{2}\Big)^2,$$

we have

$$\begin{aligned}
G(2j+1,\lambda,r) &= \sqrt{2}(2\pi)^{-\frac{2j+2}{2}} e^{-\imath(\frac{2j}{2}\pi)} \int_r^{+\infty} \frac{\sinh s}{\sqrt{\cosh s - \cosh r}}(\sinh s)^{-\frac{2j}{2}} Q^j_{\theta_n(\lambda)}(\cosh s)\,ds\\
&= \sqrt{2}(2\pi)^{-\frac{2j+2}{2}} e^{-\imath(\frac{2j}{2}\pi)} \int_{\cosh r}^{+\infty} \frac{(u^2-1)^{-\frac{j}{2}}}{\sqrt{u - \cosh r}} Q^j_{\theta_n(\lambda)}(u)\,du\\
&= (2\pi)^{-\frac{n}{2}}(\sinh r)^{-\frac{n-2}{2}} e^{-\imath(\pi\frac{n-2}{2})} Q^{\frac{n-2}{2}}_{\theta_n(\lambda)}(\cosh r),
\end{aligned}$$

where the last equality comes from the formula §7.133 2 of [23] (p. 773). ∎

4.3 A lower estimate of $G(n, -\varpi^2, r)$

In the following, we write

$$\sqrt{\rho^2 - \varpi^2} = \alpha\rho, \quad \text{with } 0 < \alpha < 1. \tag{4.12}$$

Then, we have $\varpi^2 = (1-\alpha^2)\rho^2$ and $\theta_n(-\varpi^2) = \alpha\rho - \frac{1}{2}$.

We have the following lower estimate for $G(n, -(1-\alpha^2)\rho^2, r)$, which will be crucial for this paper:

Lemma 4.1 *For $n \geq 3$ and $0 < \alpha < 1$ satisfying $\alpha\rho > \frac{1}{2}$ and $(1-\alpha)\rho \geq 1$, we have*

$$G(n, -(1-\alpha^2)\rho^2, r) \geq \frac{1}{n(n-2)} \frac{1}{\Omega_n (\sinh r)^{n-2}} \Big(\cosh \frac{r}{2} \Big)^{2\rho(1-\alpha)-2}, \quad \forall r > 0. \tag{4.13}$$

Proof. For $n \geq 3$ and $0 < \alpha < 1$ satisfying $\alpha\rho > \frac{1}{2}$, (4.9) and (4.11) imply that

$$\begin{aligned} & G(n, -(1-\alpha^2)\rho^2, r) \\ &= (2\pi)^{-\frac{n}{2}} (\sinh r)^{2-n} 2^{-\alpha\rho - \frac{1}{2}} \frac{\Gamma(\rho(1+\alpha))}{\Gamma(\alpha\rho + \frac{1}{2})} \int_0^\pi (\cosh r + \cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt. \end{aligned}$$

Meanwhile,

$$\begin{aligned} & \int_0^\pi (\cosh r + \cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt \\ &= \int_0^\pi (1 + \cos t + 2\sinh^2 \frac{r}{2})^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt \\ &= \int_0^\pi (1+\cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \Big(1 + \frac{2\sinh^2 \frac{r}{2}}{1+\cos t} \Big)^{\rho(1-\alpha)-1} dt \\ &\geq \Big(1 + \sinh^2 \frac{r}{2} \Big)^{\rho(1-\alpha)-1} \int_0^\pi (1+\cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt \\ & \quad \text{since } (1-\alpha)\rho \geq 1 \\ &= \Big(\cosh \frac{r}{2} \Big)^{2\rho(1-\alpha)-2} \int_0^\pi (1+\cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt, \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi (1+\cos t)^{\rho(1-\alpha)-1} (\sin t)^{2\alpha\rho} \, dt &= 2 \int_0^\pi \Big(2\cos^2 \frac{t}{2} \Big)^{\rho(1-\alpha)-1} \Big(2 \sin \frac{t}{2} \cos \frac{t}{2} \Big)^{2\alpha\rho} d\frac{t}{2} \\ &= 2^{\rho(1+\alpha)} \int_0^{\frac{\pi}{2}} (\cos z)^{2\rho-2} (\sin z)^{2\alpha\rho} \, dz \\ &= 2^{\rho(1+\alpha)-1} B(\rho - \frac{1}{2}, \alpha\rho + \frac{1}{2}) \ \text{(cf.§8.380 2 of [23] p. 908)} \\ &= 2^{\rho(1+\alpha)-1} \frac{\Gamma(\rho - \frac{1}{2}) \Gamma(\alpha\rho + \frac{1}{2})}{\Gamma(\rho(1+\alpha))}. \end{aligned}$$

Hence we obtain immediately (4.13). ■

4.4 Estimate for the (micro-)part at infinity

In this paper, we will need an elementary estimate as follows.

Lemma 4.2 *Let*

$$\Phi(s) = \frac{\ln\cosh s}{s^2}, s > 0. \tag{4.14}$$

We then have

$$0 < \Phi(s_o) = \inf_{0<s\leq s_o} \frac{\ln\cosh s}{s^2} < \frac{1}{2}, \quad s_o > 0. \tag{4.15}$$

Proof. Observe that

$$\lim_{s\longrightarrow 0^+} \Phi(s) = \lim_{s\longrightarrow 0^+} \frac{\ln(1+2\sinh^2\frac{s}{2})}{s^2} = \frac{1}{2},$$
$$\Phi'(s) = s^{-3}(s\tanh s - 2\ln\cosh s) = s^{-3}\varepsilon(s),$$
$$\text{with } \varepsilon(0) = 0 \text{ and } \varepsilon'(s) = \frac{2s - \sinh(2s)}{2\cosh^2 s} < 0.$$

We have thus proved (4.15). ■

For fixed $1 < p < 2$, we write in what follows

$$p' = \frac{p}{p-1}, \qquad \alpha = p^{-\frac{1}{2}} > \frac{1}{2}. \tag{4.16}$$

Let $0 < \epsilon_o < 1$ be a constant to be determined later. For $n \geq 3$ and $(1-\alpha)\rho \geq 1$, by (4.13), we have

$$\frac{1}{\Omega_n(\sinh r)^n} \leq n(n-2)(\sinh\epsilon_o)^{-2}(\cosh\frac{\epsilon_o}{2})^{2-2(1-\alpha)\rho} G(n, -p'^{-1}\rho^2, r), \quad \forall \epsilon_o \leq r \leq 1,$$

and

$$\frac{1}{\Omega_n(\sinh r)^{n-1}} \leq n(n-2)G(n, -p'^{-1}\rho^2, r), \quad \forall r \geq 1.$$

(4.5) implies that there exists a constant $C_* > 0$, independent of n, such that

$$\begin{aligned} S_{\epsilon_o} f(g) &= \int_{d(g,\xi)\geq\epsilon_o} \frac{|f|(\xi)}{V(d(g,\xi))}\, d\mu(\xi) \\ &\leq n(n-2)C_* \max\left\{(\sinh\epsilon_o)^{-2}(\cosh\frac{\epsilon_o}{2})^{2-2(1-\alpha)\rho}, 1\right\}\left[-\frac{1}{p'}\rho^2 - \Delta_{\mathbb{H}^n}\right]^{-1}(|f|)(g). \end{aligned}$$

For $0 < s \leq 1$, we observe first that

$$(\sinh s)^{-2}(\cosh\frac{s}{2})^{2-2(1-\alpha)\rho} \leq 4s^{-2}e^{-2(1-\alpha)\rho\ln\cosh\frac{s}{2}},$$

then, by (4.15), that

$$(\sinh s)^{-2}(\cosh\frac{s}{2})^{2-2(1-\alpha)\rho} \leq 4s^{-2}e^{-2^{-1}\Phi(2^{-1})(1-\alpha)\rho s^2}.$$

We can see that

$$s^{-2}e^{-2^{-1}\Phi(2^{-1})(1-\alpha)\rho s^2} \leq 1 \Longleftrightarrow (\sqrt{\rho}s)^2 e^{2^{-1}\Phi(2^{-1})(1-\alpha)(\sqrt{\rho}s)^2} \geq \rho.$$

So, for $\rho \geq \rho_o(\alpha) \gg 1$, when

$$1 \geq s \geq \Big[2^{-1}\Phi(2^{-1})(1-\alpha)\Big]^{-\frac{1}{2}}\sqrt{\frac{\ln\rho}{\rho}},$$

we have

$$(\sinh s)^{-2}(\cosh\frac{s}{2})^{2-2(1-\alpha)\rho} \leq 4s^{-2}e^{-2^{-1}\Phi(2^{-1})(1-\alpha)\rho s^2} \leq 4.$$

As a consequence, we obtain

Proposition 4.3 *Let* $1 < p < 2$ *and*

$$n(p) = \min\Big\{n \geq 100; \Big[2^{-1}\Phi(2^{-1})(1-p^{-\frac{1}{2}})\Big]^{-\frac{1}{2}}\sqrt{\frac{\ln(\frac{n-1}{2})}{\frac{n-1}{2}}} < \frac{1}{2}\Big\},$$

where Φ *is defined by* (4.14). *Then, for all* $n \geq n(p)$ *and all*

$$1 > \epsilon_o \geq \Big[2^{-1}\Phi(2^{-1})(1-p^{-\frac{1}{2}})\Big]^{-\frac{1}{2}}\sqrt{\frac{\ln\rho}{\rho}},$$

we have for all f *and all* $g \in \mathbb{H}^n$,

$$S_{\epsilon_o}f(g) = \int_{d(g,\xi)\geq\epsilon_o}\frac{|f|(\xi)}{V(d(g,\xi))}\,d\mu(\xi) \leq 8C_* n(n-2)\Big[-\frac{1}{p'}\rho^2 - \Delta_{\mathbb{H}^n}\Big]^{-1}(|f|)(g), \quad (4.17)$$

where the constant $C_* > 0$ *comes from* (4.5).

4.5 Estimate for the (micro-)local part

Recall that $M_{\mathbb{R}^{n-1}}$ stands for the centered Hardy-Littlewood maximal function on $\mathbb{R}^{n-1}$ and that $e^{sL_{n-1}}$ $(s > 0)$ is the heat semigroup defined by the Sturm-Liouville operator $L_{n-1} = y^2\frac{d^2}{dy^2} - (n-2)y\frac{d}{dy}$ on $L^2(\mathbb{R}^+, y^{-n}\,dy)$. We have then the following

Proposition 4.4 *Let* $A > 0$ *and*

$$n(A) = \min\Big\{n \geq 100; \frac{A(n-1)^{-\frac{1}{4}}}{2} \leq c_o\Big\}, \text{ where } c_o > 0 \text{ is the same as in } (3.1).$$

Then there exists a constants $c(A) > 0$ such that for all $n \geq n(A)$ and $0 < \epsilon_o < 1$ satisfying $0 < (n-1)\epsilon_o^4 \leq A$, we have

$$\sup_{0<r<\epsilon_o} \frac{1}{|B(g,r)|} \int_{B(g,r)} |f(\xi)|\, d\mu(\xi) \leq c(A) \sup_{s>0} e^{sL_{n-1}} \Big(M_{\mathbb{R}^{n-1}} f(\cdot, x) \Big)(y), \tag{4.18}$$

for all continuous functions f and all $g = (y,x) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}$.

Proof. For $g = (y,x) \in \mathbb{H}^n$ and $0 < r \leq 1$, we have

$$\begin{aligned}
&\frac{1}{|B(g,r)|} \int_{B(g,r)} |f(\xi)|\, d\mu(\xi) \\
&\leq \frac{C_*}{\Omega_n (\sinh r)^n} \int_{ye^{-r}}^{ye^r} \Big[\int_{B_{\mathbb{R}^{n-1}}(x, \sqrt{2yv\cosh r - (y^2+v^2)})} |f(v,w)|\, dw \Big] v^{-n}\, dv \\
&\leq \frac{C_* \Omega_{n-1}}{\Omega_n (\sinh r)^n} \int_{ye^{-r}}^{ye^r} \Big[2yv\cosh r - (y^2+v^2) \Big]^{\frac{n-1}{2}} M_{\mathbb{R}^{n-1}} f(v,x)\, v^{-n}\, dv \\
&\leq c' \frac{\sqrt{n-1}}{\sinh r} \int_{ye^{-r}}^{ye^r} \Big[\frac{2yv\cosh r - (y^2+v^2)}{\sinh^2 r} \Big]^{\frac{n-1}{2}} M_{\mathbb{R}^{n-1}} f(v,x)\, v^{-n}\, dv,
\end{aligned}$$

where the last inequality comes from the explicit expression of Ω_n and the Stirling's formula.

By writing $\tau = \ln \frac{y}{v}$, we observe first that

$$\begin{aligned}
\frac{2yv\cosh r - (y^2+v^2)}{\sinh^2 r} &= yv \frac{2\cosh r - 2\cosh \tau}{\sinh^2 r} = yv \frac{\sinh^2 \frac{r}{2} - \sinh^2 \frac{\tau}{2}}{\cosh^2 \frac{r}{2} \sinh^2 \frac{r}{2}} \\
&= yv \frac{1}{\cosh^2 \frac{r}{2}} \Big(1 - \frac{\sinh^2 \frac{\tau}{2}}{\sinh^2 \frac{r}{2}} \Big),
\end{aligned}$$

and then

$$\begin{aligned}
\Big[\frac{2yv\cosh r - (y^2+v^2)}{\sinh^2 r} \Big]^{\frac{n-1}{2}} &= (yv)^{\frac{n-1}{2}} e^{-(n-1)\ln\cosh\frac{r}{2}} \exp\Big\{ \frac{n-1}{2} \ln\big(1 - \frac{\sinh^2 \frac{\tau}{2}}{\sinh^2 \frac{r}{2}}\big) \Big\} \\
&= (yv)^{\frac{n-1}{2}} e^{-(n-1)\ln\cosh\frac{r}{2}} \exp\Big\{ -\frac{n-1}{2} (\frac{\tau}{r})^2 + \frac{n-1}{2} F_{\frac{r}{2}}(\frac{\tau}{2}) \Big\},
\end{aligned}$$

where F_β is defined in Lemma 3.1.

When $n \geq n(A)$ and $0 < \epsilon_o < 1$ satisfying $0 < (n-1)\epsilon_o^4 \leq A$, we have

$$e^{-(n-1)\ln\cosh\frac{r}{2}} = e^{-(n-1)\ln(1+2\sinh^2\frac{r}{4})} = e^{-\frac{n-1}{8}r^2 + O((n-1)r^4)}, \qquad 0 < r < \epsilon_o,$$

(3.1) implies that

$$\Big[\frac{2yv\cosh r - (y^2+v^2)}{\sinh^2 r} \Big]^{\frac{n-1}{2}} \leq c'(A) (yv)^{\frac{n-1}{2}} e^{-\frac{n-1}{8}r^2 - \frac{n-1}{2}\frac{\ln^2 \frac{y}{v}}{r^2}}.$$

As a consequence, for $0 < r \leq \epsilon_o \leq A(n-1)^{-\frac{1}{4}}$ and $(y,x) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}$, we have

$$\frac{1}{|B((y,x),r)|} \int_{B((y,x),r)} |f(v,w)|\, d\mu(v,w) \leq c(A) e^{\frac{r^2}{2(n-1)} L_{n-1}} \Big(M_{\mathbb{R}^{n-1}} f(\cdot, x) \Big)(y).$$

This concludes the proof of Proposition 4.4. ■

4.6 Proof of Theorem 1.1 for real hyperbolic spaces

As $\|M\|_{L^\infty \longrightarrow L^\infty} = 1$, by the Marcinkiewicz interpolation theorem, it is sufficient to show that (1.7) is true for $1 < p < 2$.

We write in the following

$$n^*(p) = \min \Big\{ n_* \geq 100; \Big[2^{-1}\Phi(2^{-1})(1 - p^{-\frac{1}{2}})\Big]^{-\frac{1}{2}} \sqrt{\frac{\ln(2^{-1}(n-1))}{2^{-1}(n-1)}} \leq (n-1)^{-\frac{1}{4}} \leq 2c_o, \\ \forall n \geq n_* \Big\}.$$

It is well-known that

$$\|M\|_{L^p(\mathbb{H}^n) \longrightarrow L^p(\mathbb{H}^n)} \leq C(n,p), \quad \forall n \geq 2, 1 < p < 2,$$

therefore, there exists a constant $C(p) > 1$ such that

$$\|M\|_{L^p(\mathbb{H}^n) \longrightarrow L^p(\mathbb{H}^n)} \leq C(p), \quad \forall 2 \leq n \leq n^*(p), 1 < p < 2.$$

For $n > n^*(p)$, set

$$r_* = r_*(n,p) = \Big[2^{-1}\Phi(2^{-1})(1 - p^{-\frac{1}{2}})\Big]^{-1} \sqrt{\frac{\ln(2^{-1}(n-1))}{2^{-1}(n-1)}} \leq (n-1)^{-\frac{1}{4}} \leq 2c_o \ll 1.$$

We get that

$$\begin{aligned} Mf(g) &= \sup_{r>0} \frac{1}{V(r)} \int_{B(g,r)} |f(\xi)|\, d\mu(\xi) \\ &\leq \int_{d(g,\xi) \geq r_*} \frac{|f|(\xi)}{V(d(g,\xi))}\, d\mu(\xi) + \sup_{0<r\leq r_*} \frac{1}{V(r)} \int_{B(g,r)} |f(\xi)|\, d\mu(\xi) \\ &\leq 8C_* n(n-2)\Big[-\frac{1}{p'}\rho^2 - \Delta_{\mathbb{H}^n}\Big]^{-1}(|f|)(g) + c \sup_{s>0} e^{sL_{n-1}}\Big(M_{\mathbb{R}^{n-1}} f(\cdot, x)\Big)(y), \end{aligned}$$

where the last inequality follows from the Propositions 4.3 and 4.4.

We have also

$$\begin{aligned} &\Big\| n(n-2)\Big[-\frac{1}{p'}\rho^2 - \Delta_{\mathbb{H}^n}\Big]^{-1} \Big\|_{L^p(\mathbb{H}^n) \longrightarrow L^p(\mathbb{H}^n)} \\ &\leq n(n-2) \int_0^{+\infty} e^{\frac{1}{p'}\rho^2 s} \|e^{s\Delta_{\mathbb{H}^n}}\|_{L^p(\mathbb{H}^n) \longrightarrow L^p(\mathbb{H}^n)}\, ds. \end{aligned}$$

The fact that

$$\begin{aligned} \|e^{s\Delta_{\mathbb{H}^n}}\|_{L^2(\mathbb{H}^n) \longrightarrow L^2(\mathbb{H}^n)} &\leq e^{-\rho^2 s}, \quad \forall s > 0, \\ \|e^{s\Delta_{\mathbb{H}^n}}\|_{L^1(\mathbb{H}^n) \longrightarrow L^1(\mathbb{H}^n)} &\leq 1, \quad \forall s > 0, \end{aligned}$$

and the Riesz-Thorin interpolation theorem imply that

$$\|e^{s\Delta_{\mathbb{H}^n}}\|_{L^p(\mathbb{H}^n)\longrightarrow L^p(\mathbb{H}^n)} \leq e^{-\frac{2}{p'}\rho^2 s}, \quad \forall s>0, 1<p<2.$$

We have then

$$\Big\| n(n-2)\Big[-\frac{1}{p'}\rho^2-\Delta_{\mathbb{H}^n}\Big]^{-1}\Big\|_{L^p(\mathbb{H}^n)\longrightarrow L^p(\mathbb{H}^n)} \leq n(n-2)\int_0^{+\infty} e^{-\frac{1}{p'}\rho^2 s}\, ds \leq 4p'.$$

On the other hand, by using the maximal theorem for symmetric diffusion semigroups (cf. eg. [45]), we have

$$\begin{aligned}
&\Big\|\sup_{s>0} e^{sL_{n-1}}\Big(M_{\mathbb{R}^{n-1}}f(\cdot,x)\Big)(y)\Big\|^p_{L^p(\mathbb{H}^n)}\\
&=\int_{\mathbb{R}^{n-1}}\Big\{\int_0^{+\infty}\sup_{s>0}\Big[e^{sL_{n-1}}\Big(M_{\mathbb{R}^{n-1}}f(\cdot,x)\Big)(y)\Big]^p y^{-n}dy\Big\}\,dx\\
&\leq C_p\int_{\mathbb{R}^{n-1}}\Big\{\int_0^{+\infty}\Big(M_{\mathbb{R}^{n-1}}f(v,x)\Big)^p v^{-n}dv\Big\}\,dx\\
&=C_p\int_0^{+\infty}\Big\{\int_{\mathbb{R}^{n-1}}\Big(M_{\mathbb{R}^{n-1}}f(v,x)\Big)^p dx\Big\}\,v^{-n}dv\\
&\leq C'_p\int_0^{+\infty}\Big\{\int_{\mathbb{R}^{n-1}}|f(v,w)|^p\,dw\Big\}\,v^{-n}dv \quad \text{by (1.2)}\\
&=C'_p\|f\|^p_{L^p(\mathbb{H}^n)}.
\end{aligned}$$

Hence the claim is proved. ■

5 The case of complex hyperbolic spaces

The goal of this section is to prove Theorem 1.1 for complex hyperbolic spaces.

5.1 Notations and estimates of ball volume

In order to see clearly how we can adapt the method of this paper for harmonic AN groups, we consider the complex hyperbolic space of dimension $2n$ ($n\geq 2$), $\mathbb{H}^n_c$, as the group $\mathbb{R}^+\times \mathrm{H}(2(n-1),1)$ where $\mathrm{H}(2(n-1),1)$ stands for the Heisenberg group of dimension $2n-1$. Recall that $\mathrm{H}(2(n-1),1)=\mathbb{C}^{n-1}\times\mathbb{R}$ is a stratified Lie group with the group law

$$(x,\varrho)\cdot(w,u)=(x+w,\varrho+u+2^{-1}\langle x,Uw\rangle),$$

where

$$\langle x,Uw\rangle=\Im\langle x,w\rangle,\quad x=(z_1,\ldots,z_{n-1}),\quad w=(z'_1,\ldots,z'_{n-1})\in\mathbb{C}^{n-1},$$

$$z_j=x_j+\imath y_j\ (x_j,y_j\in\mathbb{R}),\quad \langle x,w\rangle=\sum_{j=1}^{n-1}z_j\cdot\overline{z'_j}.$$

In what follows, we denote $o=(0,0)$ the origin of $\mathrm{H}(2(n-1),1)$, $(x,\varrho)\in\mathbb{C}^{n-1}\times\mathbb{R}$ a point of $\mathrm{H}(2(n-1),1)$, and set $|x|^2=\sum_{j=1}^{n-1}\|z_j\|^2$. We recall that the Haar measure on $\mathrm{H}(2(n-1),1)$ is that of Lebesgue.

The canonical sub-Laplacian on $\mathrm{H}(2(n-1),1)$ can be written as

$$\Delta_{\mathrm{H}(2(n-1),1)}=\sum_{j=1}^{n-1}(\mathrm{X}_j^2+\mathrm{Y}_j^2),$$

where X_j and Y_j $(1\le j\le n-1)$ are the left-invariant vector fields defined by

$$\mathrm{X}_j=\frac{\partial}{\partial x_j}+\frac{1}{2}y_j\frac{\partial}{\partial\varrho},\qquad \mathrm{Y}_j=\frac{\partial}{\partial y_j}-\frac{1}{2}x_j\frac{\partial}{\partial\varrho}.$$

We write also $\mathrm{T}=\frac{\partial}{\partial\varrho}$. Recall that

$$(x,\varrho)^{-1}=(-x,-\varrho),\qquad \delta_r(x,\varrho)=(rx,r^2\varrho),\forall r>0.$$

The multiplication law on $\mathbb{R}^+\times\mathrm{H}(2(n-1),1)$ is:

$$(a,(x,\varrho))\cdot(h,(w,u))=(ah,(x,\varrho)\cdot\delta_{\sqrt{a}}(w,u)).$$

We have then

$$(a,(x,\varrho))^{-1}=(a^{-1},\delta_{a^{-\frac{1}{2}}}((x,\varrho)^{-1})).$$

The Laplacian on $\mathbb{R}^+\times\mathrm{H}(2(n-1),1)$ is (cf. eg. [15] or [18])

$$\Delta_{\mathbb{H}_c^n}=a^2\frac{\partial^2}{\partial a^2}-(n-1)a\frac{\partial}{\partial a}+a\Delta_{\mathrm{H}(2(n-1),1)}+a^2\mathrm{T},$$

and the spectral gap of $-\Delta_{\mathbb{H}_c^n}$ on $L^2(\mathbb{H}_c^n)$ is (cf. eg. [3])

$$\rho_c^2=\frac{n^2}{4}. \tag{5.1}$$

Denote $e=(1,o)$ the identity element, the induced distance between e and $(a,(x,\varrho))$ can be written as (cf.(2.18) of [3]):

$$\cosh d((a,(x,\varrho)),e)=\frac{(\frac{|x|^2}{4})^2+|\varrho|^2+(1+a^2)+\frac{|x|^2}{2}(1+a)}{2a}. \tag{5.2}$$

Observe that for $g=(a,(x,\varrho))$ and $\xi=(h,(w,u))$, we have

$$g^{-1}\xi=(a^{-1},\delta_{a^{-\frac{1}{2}}}((x,\varrho)^{-1}))(h,(w,u))=(\frac{h}{a},\delta_{a^{-\frac{1}{2}}}((x,\varrho)^{-1}\cdot(w,u))),$$
$$d(g,\xi)=d(g^{-1}\xi,e). \tag{5.3}$$

We get that the open ball $B((a,(x,\varrho)),r)$ with center $g=(a,(x,\varrho))$ and radius $r>0$ is the set

$$
\begin{aligned}
&B((a,(x,\varrho)),r)\\
&=\Big\{\xi=(h,(w,u));e^{-r}<\frac{h}{a}<e^{r},\\
&\quad\frac{|x-w|^2}{2a}(1+\frac{h}{a})+\frac{|x-w|^4}{16a^2}+\Big|\frac{u}{a}-\frac{\varrho}{a}-\frac{\langle x,Uw\rangle}{2a}\Big|^2<2\frac{h}{a}\cosh r-[1+\frac{h^2}{a^2}]\Big\}. \quad (5.4)
\end{aligned}
$$

Recall that the induced measure is $d\lambda(a,(x,\varrho))=a^{-n-1}dadxd\varrho$.

We give now the estimates of ball volumes in $\mathbb{H}_c^n$. Since $|B(g,r)|$ does not depend on $g\in\mathbb{H}_c^n$, we write in what follows $V_c(r)=|B(g,r)|$. By (1.4) of [44] or (1.16) of [3], we have:

$$
\begin{aligned}
V_c(r)&=2^{2n-1}\omega_{2n-1}\int_0^r\Big(\sinh\frac{s}{2}\Big)^{2n-1}\cosh\frac{s}{2}\,ds=2^{2n}\omega_{2n-1}\frac{1}{2n}\Big(\sinh\frac{r}{2}\Big)^{2n}\\
&=2^{2n}\Omega_{2n}\Big(\sinh\frac{r}{2}\Big)^{2n}. \quad (5.5)
\end{aligned}
$$

5.2 Recall on the heat kernel and a lower estimate of the Green function

The heat kernel on $\mathbb{H}_c^n$, $K_n^c(t,g,\xi)$, is a function of $(t,d(g,\xi))$, we define $K_n^c(t,r)$ ($t>0$, $r\geq 0$) as

$$
K_n^c(t,\varsigma)=K_n^c(t,g,\xi),\qquad \text{with } \varsigma=d(g,\xi).
$$

We know that (cf. eg. (5.8) of [3]):

$$
\begin{aligned}
K_n^c(t,r)&=2^{-2n+\frac12}\pi^{-\frac{2n+1}{2}}t^{-\frac12}e^{-\rho_c^2t}\\
&\quad\times\int_r^{+\infty}\frac{\sinh s}{\sqrt{\cosh s-\cosh r}}\Big(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\Big)\Big(-\frac{1}{\sinh\frac{s}{2}}\frac{\partial}{\partial s}\Big)^{n-1}e^{-\frac{s^2}{4t}}\,ds\\
&=2^{-2n+\frac12}\pi^{-n}e^{-\rho_c^2t}\int_r^{+\infty}\frac{\sinh\frac{s}{2}}{\sqrt{\cosh s-\cosh r}}\Big(-\frac{1}{\sinh\frac{s}{2}}\frac{\partial}{\partial s}\Big)^n\Big[\frac{1}{\sqrt{\pi t}}e^{-\frac{s^2}{4t}}\Big]\,ds.
\end{aligned}
$$

And by (4.6), we have

$$
\begin{aligned}
\Big(-\frac{1}{\sinh\frac{s}{2}}\frac{\partial}{\partial s}\Big)^n\Big[\frac{1}{\sqrt{\pi t}}e^{-\frac{s^2}{4t}}\Big]&=\pi^n\Big(-\frac{1}{2\pi}\frac{\partial}{\partial\phi}\Big)^n\Big|_{\phi=\cosh\frac{s}{2}}K_1(\frac{t}{4},\operatorname{arccosh}\phi)\\
&=\pi^ne^{\frac{n^2}{4}t}K_{2n+1}(\frac{t}{4},\frac{s}{2}). \quad (5.6)
\end{aligned}
$$

So we can write

$$
K_n^c(t,r)=2^{-2n}\int_r^{+\infty}\frac{\sinh\frac{s}{2}}{\sqrt{\sinh^2\frac{s}{2}-\sinh^2\frac{r}{2}}}K_{2n+1}(\frac{t}{4},\frac{s}{2})\,ds.
$$

Notice that this type of formulae has been obtained in [35] or in [36] (but with some change of variables).

As a consequence, for $\lambda > -\rho_c^2 = -\frac{n^2}{4}$, we have

$$\begin{aligned}(\lambda - \Delta_{\mathbb{H}_c^n})^{-1}(g,\xi) &= \int_0^{+\infty} e^{-\lambda t} K_n^c(t, d(g,\xi))\, dt \\ &= 8 \times 2^{-2n} \int_{\frac{d(g,\xi)}{2}}^{+\infty} \frac{\sinh r}{\sqrt{\sinh^2 r - \sinh^2 \frac{d(g,\xi)}{2}}} G(2n+1, 4\lambda, r)\, dr.\end{aligned}$$

As in the case of real hyperbolic spaces, we have a lower estimate of $(\lambda - \Delta_{\mathbb{H}_c^n})^{-1}$ as follows:

Lemma 5.1 *For $n \geq 2$ and $0 < \alpha < 1$ satisfying $\alpha\rho_c > \frac{1}{4}$ and $(1-\alpha)\rho_c \geq 1$, we have for all $d(g,\xi) = \varsigma > 0$,*

$$\Big[-(1-\alpha^2)\rho_c^2 - \Delta_{\mathbb{H}_c^n} \Big]^{-1}(g,\xi) \geq \frac{1}{2n(2n-2)} \frac{1}{2^{2n}\Omega_{2n}(\sinh\frac{\varsigma}{2})^{2n-2}} \Big(\cosh\frac{\varsigma}{4} \Big)^{2(1-\alpha)n-4}. \quad (5.7)$$

Proof. By (4.13), first we have

$$\begin{aligned}&\Big[-(1-\alpha^2)\rho_c^2 - \Delta_{\mathbb{H}_c^n} \Big]^{-1}(g,\xi) \\ &\geq 8 \cdot 2^{-2n} \int_{\frac{\varsigma}{2}}^{+\infty} \frac{\sinh r}{\sqrt{\sinh^2 r - \sinh^2\frac{\varsigma}{2}}} \frac{(\cosh\frac{r}{2})^{2(1-\alpha)n-2}}{(2n+1)(2n-1)\Omega_{2n+1}(\sinh r)^{2n-1}}\, dr \\ &= \frac{1}{(n-\frac{1}{2})(n+\frac{1}{2})} \frac{1}{2^{2n}\Omega_{2n}} \frac{\Omega_{2n}}{\Omega_{2n+1}} \int_{\frac{\varsigma}{2}}^{+\infty} \frac{(\cosh\frac{r}{2})^{2(1-\alpha)n-4}\cosh r}{(\sinh r)^{2n-2}\sqrt{\sinh^2 r - \sinh^2\frac{\varsigma}{2}}} \frac{2(\cosh\frac{r}{2})^2}{\cosh r}\, dr,\end{aligned}$$

then, due to $(1-\alpha)\rho_c \geq 1$, we get

$$\begin{aligned}&\Big[-(1-\alpha^2)\rho_c^2 - \Delta_{\mathbb{H}_c^n} \Big]^{-1}(g,\xi) \\ &\geq \frac{\Big(\cosh\frac{\varsigma}{4}\Big)^{2(1-\alpha)n-4}}{(n-\frac{1}{2})(n+\frac{1}{2})2^{2n}\Omega_{2n}} \frac{\Omega_{2n}}{\Omega_{2n+1}} \int_{\frac{\varsigma}{2}}^{+\infty} \frac{\cosh r}{(\sinh r)^{2n-2}\sqrt{\sinh^2 r - \sinh^2\frac{\varsigma}{2}}}\, dr.\end{aligned}$$

Using the change of variables $\sqrt{\sinh^2 r - \sinh^2 \frac{\varsigma}{2}} = \sqrt{s}\sinh\frac{\varsigma}{2}$, we have

$$\begin{aligned}
&\int_{\frac{\varsigma}{2}}^{+\infty} \frac{\cosh r}{(\sinh r)^{2n-2}\sqrt{\sinh^2 r - \sinh^2 \frac{\varsigma}{2}}}\, dr \\
&= 2^{-1}\Big(\sinh\frac{\varsigma}{2}\Big)^{2-2n}\int_0^{+\infty}(1+s)^{-\frac{2n-1}{2}}s^{-\frac{1}{2}}\, ds \\
&= 2^{-1}\Big(\sinh\frac{\varsigma}{2}\Big)^{2-2n}B(\frac{1}{2}, \frac{2n-1}{2}-\frac{1}{2}) \quad \text{(cf. §3.194 3 de [23] p. 315)} \\
&= 2^{-1}\Big(\sinh\frac{\varsigma}{2}\Big)^{2-2n}\frac{\Gamma(\frac{1}{2})\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}.
\end{aligned}$$

It follows from the explicit formula of Ω_k (cf.(4.4)),

$$\begin{aligned}
&\Big[-(1-\alpha^2)\rho_c^2 - \Delta_{\mathbb{H}_c^n}\Big]^{-1}(g,\xi) \\
&\geq \frac{1}{(n-\frac{1}{2})(n+\frac{1}{2})2^{2n}\Omega_{2n}(\sinh\frac{\varsigma}{2})^{2n-2}}\Big(\cosh\frac{\varsigma}{4}\Big)^{2(1-\alpha)n-4}\frac{1}{2}\frac{\Gamma(n-1)}{\Gamma(n+1)}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n-\frac{1}{2})} \\
&= \frac{1}{2}\frac{1}{(n-1)n2^{2n}\Omega_{2n}(\sinh\frac{\varsigma}{2})^{2n-2}}\Big(\cosh\frac{\varsigma}{4}\Big)^{2(1-\alpha)n-4}.
\end{aligned}$$

Hence the desired result follows. ■

5.3 Estimate for the (micro-)part at infinity

We keep the notations of Subsection 4.4. By arguing the same way as in the proof of Proposition 4.3, (5.5) and (5.7) imply then the following proposition:

Proposition 5.2 *For $1 < p < 2$ and*

$$n(p) = \min\Big\{n \geq 100; \Big[4^{-1}\Phi(4^{-1})(1-p^{-\frac{1}{2}})\Big]^{-\frac{1}{4}}\sqrt{\frac{\ln(\frac{n}{2})}{\frac{n}{2}}} < \frac{1}{2}\Big\},$$

where Φ is defined by (4.14). Then, for all $n \geq n(p)$ and all

$$1 > \epsilon_o \geq \Big[4^{-1}\Phi(4^{-1})(1-p^{-\frac{1}{2}})\Big]^{-\frac{1}{2}}\sqrt{\frac{\ln\rho_c}{\rho_c}},$$

we have for all f and all $g \in \mathbb{H}_c^n$,

$$S_{\epsilon_o}f(g) = \int_{d(g,\xi)\geq\epsilon_o}\frac{|f|(\xi)}{V_c(d(g,\xi))}\, d\lambda(\xi) \leq 10^2 2n(2n-2)\Big[-\frac{1}{p'}\rho_c^2 - \Delta_{\mathbb{H}_c^n}\Big]^{-1}(|f|)(g). \quad (5.8)$$

5.4 Estimate for the (micro-)local part

We denote in the following $S_{\mathrm{H}(2(n-1),1)}$ the spherical maximal function on $\mathrm{H}(2(n-1),1)$, which is defined for continuous function ψ and $(x,\varrho) \in \mathrm{H}(2(n-1),1)$ by

$$S_{\mathrm{H}(2(n-1),1)}\psi(x,\varrho) = \sup_{r>0} \frac{1}{\omega_{2(n-1)-1}} \int_{\theta \in S^{2(n-1)-1}} |\psi|\Big((x,\varrho)\cdot \delta_r(\theta,0)\Big)\, d\sigma(\theta),$$

where $S^{2(n-1)-1}$ stands for the unit sphere in $\mathbb{R}^{2(n-1)}$ and $d\sigma$ its standard measure.

Proposition 5.3 *Let $A > 0$ and*

$$n(A) = \min\Big\{n \geq 100; \frac{A(n-\frac{1}{2})^{-\frac{1}{4}}}{2} \leq c_o\Big\}, \text{ where } c_o > 0 \text{ is the same as in (3.1).}$$

Then, there exists a constant $c(A) > 0$ such that for all $n \geq n(A)$ and $0 < \epsilon_o < 1$ satisfying $0 < (n - 2^{-1})\epsilon_o^4 \leq A$, we have

$$\sup_{0<r<\epsilon_o} \frac{1}{|B(g,r)|} \int_{B(g,r)} |f(\xi)|\, d\lambda(\xi) \leq c(A) \sup_{s>0} e^{sL_n}\Big\{M_{\mathbb{R}}\Big[S_{\mathrm{H}(2(n-1),1)} f(*,(x,\cdot))\Big](\varrho)\Big\}(a), \tag{5.9}$$

for all continuous functions f, on $\mathbb{R}^+ \times \mathrm{H}(2(n-1),1)$, and all $g = (a,(x,\varrho)) \in \mathbb{R}^+ \times \mathrm{H}(2(n-1),1)$.

To prove the above proposition, we need the following notations and lemmas:
For $a, h, r > 0$ with $e^{-r} < \frac{h}{a} < e^r$, set

$$\kappa = \kappa(a,h,r) = 2\frac{h}{a}\cosh r - \Big(1 + \frac{h^2}{a^2}\Big), \tag{5.10}$$

$$E_{s,\gamma} = \{(w,u); \frac{|w|^2}{2}(1+\gamma) + \frac{|w|^4}{16} + |u|^2 < s\}, \qquad \forall s, \gamma > 0. \tag{5.11}$$

For $g = (a,(x,\varrho))$, by (5.4), we have then

$$B(g,r) = \Big\{(h,(w,u)); e^{-r} < \frac{h}{a} < e^r, (x,\varrho)^{-1}\cdot(w,u) \in \delta_{\sqrt{a}}\Big(E_{\kappa,\frac{h}{a}}\Big)\Big\}. \tag{5.12}$$

We have the following two lemmas, which will be proven in Subsections 5.5 and 5.6 respectively.

Lemma 5.4 *For all continuous functions φ on $\mathrm{H}(2(n-1),1)$, and all $(x,\varrho) \in \mathrm{H}(2(n-1),1)$, we have*

$$\frac{1}{|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})|} \int_{(w,u)\in\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})} |\varphi|((x,\varrho+u)\cdot(w,0))\, dwdu \leq M_{\mathbb{R}}\Big(S_{\mathrm{H}(2(n-1),1)}\varphi(x,\cdot)\Big)(\varrho). \tag{5.13}$$

Lemma 5.5 *Let $A>0$. There exists a constant $C(A)>0$ such that for $n\geq n(A)$ and $|\tau=\ln\frac{h}{a}|<r\leq A(n-\frac{1}{2})^{-\frac{1}{4}}$, we have*

$$\frac{|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})|}{\Omega_{2n}(2\sinh\frac{r}{2})^{2(n-\frac{1}{2})}}\leq C(A)\sqrt{n-1}(ah)^{\frac{n}{2}}e^{-(n-1)\frac{\tau^2}{r^2}}e^{-\frac{n^2}{16}\frac{r^2}{n-1}}. \tag{5.14}$$

Let us first admit the above two lemmas hold and give *the proof of Proposition 5.3.* It follows from (5.5),

$$\begin{aligned}
&\frac{1}{V_c(r)}\int_{B(g,r)}|f|(\xi)\,d\lambda(\xi)\\
&=\frac{1}{2^{2n}\Omega_{2n}(\sinh\frac{r}{2})^{2n}}\int_{ae^{-r}}^{ae^r}\int_{(x,\varrho)^{-1}(w,u)\in\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})}|f|\Big(h,(w,u)\Big)h^{-n-1}\,dhdwdu\\
&=\frac{1}{2^{2n}\Omega_{2n}(\sinh\frac{r}{2})^{2n}}\int_{ae^{-r}}^{ae^r}\int_{(w,u)\in\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})}|f|\Big(h,(x,\varrho)\cdot(w,u)\Big)h^{-n-1}\,dhdwdu\\
&=\frac{1}{2^{2n}\Omega_{2n}(\sinh\frac{r}{2})^{2n}}\int_{ae^{-r}}^{ae^r}\Big[\int_{(w,u)\in\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})}|f|\Big(h,(x,\varrho+u)\cdot(w,0)\Big)\,dwdu\Big]h^{-n-1}\,dh.
\end{aligned}$$

By (5.13) and (5.14), we obtain that

$$\begin{aligned}
&\frac{1}{V_c(r)}\int_{B(g,r)}|f|(\xi)\,d\lambda(\xi)\\
&\leq c(A)\frac{\sqrt{n-1}}{2\sinh\frac{r}{2}}\int_{ae^{-r}}^{ae^r}M_{\mathbb{R}}\Big(S_{\mathrm{H}(2(n-1),1)}f(h,(x,\cdot))\Big)(\varrho)(ah)^{\frac{n}{2}}e^{-(n-1)\frac{\ln^2\frac{h}{a}}{r^2}}e^{-\frac{n^2}{16}\frac{r^2}{n-1}}\,\frac{dh}{h^{n+1}},
\end{aligned}$$

then, by (2.1) with $t=\frac{1}{4}\frac{r^2}{n-1}$, we have that

$$\frac{1}{V_c(r)}\int_{B(g,r)}|f|(\xi)\,d\lambda(\xi)\leq c(A)e^{\frac{1}{4}\frac{r^2}{n-1}L_n}\Big\{M_{\mathbb{R}}\Big[S_{\mathrm{H}(2(n-1),1)}f(*,(x,\cdot))\Big](\varrho)\Big\}(a).$$

The proof of the proposition is thus finished. ∎

5.5 Proof of (5.13)

It is sufficient to modify the proof of Lemma 4 in [54] slightly. The main idea is to use the following elementary property of $M_{\mathbb{R}^m}$: for suitable f, we have

$$|f*\phi|(u)\leq\|\phi\|_1M_{\mathbb{R}^m}f(u),\quad\forall u\in\mathbb{R}^m,$$

where $\phi\geq0$ is an integrable radially decreasing function on $\mathbb{R}^m$. ∎

5.6 Proof of (5.14)

First we have

$$
\begin{aligned}
&|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})| \\
&= a^n \int_{\frac{|w|^2}{2}(1+\frac{h}{a})+\frac{|w|^4}{16}+|u|^2<\kappa} dwdu \\
&= 2a^n \int_{\frac{|w|^2}{4}+(1+\frac{h}{a})<\sqrt{\kappa+(1+\frac{h}{a})^2}} \sqrt{\kappa+(1+\frac{h}{a})^2-\Big[\frac{|w|^2}{4}+(1+\frac{h}{a})\Big]^2}\,dw \\
&\le 4a^n\Big[\kappa+(1+\frac{h}{a})^2\Big]^{\frac14}\int_{\frac{|w|^2}{4}+(1+\frac{h}{a})<\sqrt{\kappa+(1+\frac{h}{a})^2}} \sqrt{\sqrt{\kappa+(1+\frac{h}{a})^2}-\Big[\frac{|w|^2}{4}+(1+\frac{h}{a})\Big]}\,dw \\
&= 2\Big\{4\Big[\sqrt{\kappa+(1+\frac{h}{a})^2}-(1+\frac{h}{a})\Big]\Big\}^{n-\frac12} a^n\Big[\kappa+(1+\frac{h}{a})^2\Big]^{\frac14}\int_{B_{\mathbb{R}^{2(n-1)}}(o,1)}\sqrt{1-|w|^2}\,dw \\
&= B(n-1,\frac32)\omega_{2(n-1)-1}a^n\Big\{4\Big[\sqrt{\kappa+(1+\frac{h}{a})^2}-(1+\frac{h}{a})\Big]\Big\}^{n-\frac12}\Big[\kappa+(1+\frac{h}{a})^2\Big]^{\frac14} \\
&= \frac{\Gamma(n+1)}{2\sqrt{\pi}\Gamma(n+\frac12)}2^{2n}\Omega_{2n}a^n\Big[\sqrt{\kappa+(1+\frac{h}{a})^2}-(1+\frac{h}{a})\Big]^{n-\frac12}\Big[\kappa+(1+\frac{h}{a})^2\Big]^{\frac14}.
\end{aligned}
$$

For $0<r\le 1$ and $e^{-r}<\frac{h}{a}=e^{\tau}<e^{r}$, it is easy to get

$$
|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})| \le 100\frac{\Gamma(n+1)}{\Gamma(n+\frac12)}2^{2n-1}\Omega_{2n}a^n\Big[\sqrt{\kappa+(1+\frac{h}{a})^2}-(1+\frac{h}{a})\Big]^{n-\frac12},
$$

and

$$
\begin{aligned}
&2^{2n-1}a^n\Big[\sqrt{\kappa+(1+\frac{h}{a})^2}-(1+\frac{h}{a})\Big]^{n-\frac12} \\
&= a^n\kappa^{n-\frac12}\Big[\frac{\sqrt{\kappa+(1+\frac{h}{a})^2}+(1+\frac{h}{a})}{4}\Big]^{-(n-\frac12)} \\
&= a^n\Big[2\frac{h}{a}\cosh r-\Big(1+\frac{h^2}{a^2}\Big)\Big]^{n-\frac12}\Big[\frac{2\sqrt{\frac{h}{a}}\cosh\frac{r}{2}+(1+\frac{h}{a})}{4}\Big]^{-(n-\frac12)} \\
&= (ah)^{\frac{n}{2}}\Big(\frac{h}{a}\Big)^{-\frac14}\Big[2\cosh r-2\cosh\tau\Big]^{n-\frac12}\Big[\frac{\cosh\frac{r}{2}+\cosh\frac{\tau}{2}}{2}\Big]^{-(n-\frac12)} \\
&\le 2(ah)^{\frac{n}{2}}\Big[4(\sinh^2\frac{r}{2}-\sinh^2\frac{\tau}{2})\Big]^{n-\frac12}\Big[1+\sinh^2\frac{r}{4}+\sinh^2\frac{\tau}{4}\Big]^{-(n-\frac12)}.
\end{aligned}
$$

Hence,

$$
\frac{|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})|}{\Omega_{2n}(2\sinh\frac{r}{2})^{2(n-\frac12)}} \le 200\frac{\Gamma(n+1)}{\Gamma(n+\frac12)}(ah)^{\frac{n}{2}}e^{(n-\frac12)\ln(1-\frac{\sinh^2\frac{\tau}{2}}{\sinh^2\frac{r}{2}})}\Big[1+\sinh^2\frac{r}{4}+\sinh^2\frac{\tau}{4}\Big]^{-(n-\frac12)}.
$$

But there exists a constant $c(A)>0$ such that for $n\geq n(A)$ and $|\tau|<r\leq A(n-\frac{1}{2})^{-\frac{1}{4}}$, we have

$$\begin{aligned}\Big[1+\sinh^2\frac{r}{4}+\sinh^2\frac{\tau}{4}\Big]^{-(n-\frac{1}{2})} &= \Big[1+\sinh^2\frac{r}{4}\Big]^{-(n-\frac{1}{2})}\Big[1+\frac{\sinh^2\frac{\tau}{4}}{\cosh^2\frac{r}{4}}\Big]^{-(n-\frac{1}{2})}\\ &\leq \Big[1+\sinh^2\frac{r}{4}\Big]^{-(n-\frac{1}{2})}\Big[1+\frac{\sinh^2\frac{\tau}{4}}{2}\Big]^{-(n-\frac{1}{2})}\\ &\leq c(A)e^{-\frac{n-\frac{1}{2}}{16}r^2}e^{-\frac{n-\frac{1}{2}}{32}\tau^2},\end{aligned}$$

and (3.1) implies that

$$e^{(n-\frac{1}{2})\ln(1-\frac{\sinh^2\frac{\tau}{2}}{\sinh^2\frac{r}{2}})}\leq c(A)e^{-(n-\frac{1}{2})\frac{\tau^2}{r^2}}.$$

As a consequence, we have

$$\frac{|\delta_{\sqrt{a}}(E_{\kappa,\frac{h}{a}})|}{\Omega_{2n}(2\sinh\frac{r}{2})^{2(n-\frac{1}{2})}}\leq C(A)\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}(ah)^{\frac{n}{2}}e^{-(n-1)\frac{\tau^2}{r^2}}e^{-\frac{n^2}{16}\frac{1}{n-1}r^2}.$$

By using Stirling's formula, we obtain immediately (5.14). ■

5.7 Proof of the Theorem 1.1 for complex hyperbolic spaces

It suffices to prove it in the same way as the case of real hyperbolic spaces, and use the fact that (cf. Lemma 5 of [54]):

$$\Big\|S_{\mathrm{H}(2(n-1),1)}\Big\|_{L^p(\mathrm{H}(2(n-1),1))\longrightarrow L^p(\mathrm{H}(2(n-1),1))}\leq C(p),\quad \forall n\geq n_*(p).$$

6 The general case of harmonic AN groups

There are a lot of work on harmonic AN groups, cf. for example [15]-[17], [13], [18], [44], [3] and the references therein. As we have seen in the cases of real and complex hyperbolic spaces, we only use a few properties: the multiplication law and the distance formula, the induced measure and the estimates of ball volumes, the spectral gap of the Laplacian as well as the explicit expression for the heat kernel. In the sequel, we briefly recall the notations that we need.

First recall the definition of H-type group. To simplify the notations, we will use the equivalent definition in [5] (Theorem A.2, p. 199), and we refer to [25] for the original definition. An H-type group can be considered as $\mathrm{H}(2n,m)=\mathbb{R}^{2n}\times\mathbb{R}^m$ $(m,n\in\mathbb{N}^*)$ equipped with the group law

$$(x,\varrho)\cdot(w,u)=(x+w,\varrho+u+2^{-1}\langle x,Uw\rangle),$$

with $w, x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, $u, \varrho = (\varrho_1, \ldots, \varrho_m) \in \mathbb{R}^m$ and

$$\langle x, Uw \rangle = (\langle x, U^{(1)} w \rangle, \ldots, \langle x, U^{(m)} w \rangle) \in \mathbb{R}^m,$$

where the matrices $U^{(1)}$, ..., $U^{(m)}$ satisfy the following two conditions:
1. $U^{(j)}$ is a $(2n) \times (2n)$ skew-symmetric and orthogonal matrix, for all $1 \leq j \leq m$.
2. $U^{(i)} U^{(j)} + U^{(j)} U^{(i)} = 0$ for all $1 \leq i \neq j \leq m$.

Let $U^{(j)} = (U^{(j)}_{k,l})_{k,l \leq 2n}$ $(1 \leq j \leq m)$. The canonical sub-Laplacian on $\mathbb{H}(2n, m)$ can be written as $\Delta = \sum_{l=1}^{2n} X_l^2$, where X_l $(1 \leq l \leq 2n)$ are the left-invariant vector fields on $\mathbb{H}(2n, m)$, defined by

$$\mathrm{X}_l = \frac{\partial}{\partial x_l} + \frac{1}{2} \sum_{j=1}^{m} \Big(\sum_{k=1}^{2n} x_k U^{(j)}_{k,l} \Big) \frac{\partial}{\partial \varrho_j}.$$

Denote also $\mathrm{T}_j = \frac{\partial}{\partial \varrho_j}$ $(1 \leq j \leq m)$.

We recall, cf.[25], that m can be arbitrary and that $(2n, m)$ must satisfy the following condition: let $2n = (2l+1) 2^{4p+q}$ for some $l, p \in \mathbb{N}$ and $0 \leq q < 3$, then

$$m < \rho(2n) = 8p + 2^q.$$

We know that

$$(x, \varrho)^{-1} = (-x, -\varrho), \qquad \delta_r(x, \varrho) = (rx, r^2 \varrho), \forall r > 0.$$

A harmonic AN group of base $\mathrm{H}(2n, m)$ can then be considered as $\mathbb{R}^+ \times \mathrm{H}(2n, m)$ equipped with the group law

$$(a, (x, \varrho)) \cdot (h, (w, u)) = (ah, (x, \varrho) \cdot \delta_{\sqrt{a}}(w, u)).$$

Denote in what follows

$$Q = n + m, \quad |x|^2 = \sum_{k=1}^{2n} x_k^2, \quad |\varrho|^2 = \sum_{j=1}^{m} \varrho_j^2.$$

The Laplacian on $\mathbb{R}^+ \times \mathrm{H}(2n, m)$ can be written as (cf. eg. [15] or [18])

$$\Delta_{\mathbb{R}^+ \times \mathrm{H}(2n,m)} = a^2 \frac{\partial^2}{\partial a^2} - (Q-1) a \frac{\partial}{\partial a} + a \Delta_{\mathrm{H}(2n,m)} + a^2 \sum_{j=1}^{m} \mathrm{T_j}^2,$$

and the spectral gap of $-\Delta_{\mathbb{R}^+ \times \mathrm{H}(2n,m)}$ on $L^2(\mathbb{R}^+ \times \mathrm{H}(2n, m))$ is (cf. eg. [3])

$$\rho^2_{\mathbb{R}^+ \times \mathrm{H}(2n,m)} = \frac{Q^2}{4}.$$

We observe that (5.2), (5.3) and (5.4) remain valid in the case of $\mathbb{R}^+ \times \mathrm{H}(2n, m)$. The induced measure is $d\lambda(a, (x, \varrho)) = a^{-Q-1} da dx d\varrho$. We remark that $|B(g, r)|$ does not

depend on g. Define $V_{\mathbb{R}^+\times \mathrm{H}(2n,m)}(r) = |B(g,r)|$, then we have (cf. eg. (1.4) of [44] or (1.16) of [3]):

$$V_{\mathbb{R}^+\times \mathrm{H}(2n,m)}(r) = 2^{2n+m}\omega_{2n+m}\int_0^r \Big(\sinh\frac{s}{2}\Big)^{2n+m}\Big(\cosh\frac{s}{2}\Big)^m\,ds.$$

We can easily obtain the following

Lemma 6.1 *There exist two constants $c, C > 0$ such that*

$$c \le \frac{V_{\mathbb{R}^+\times \mathrm{H}(2n,m)}(r)}{2^{2n+m+1}\Omega_{2n+m+1}\Big(\sinh\frac{r}{2}\Big)^{2n+m+1}\Big(\cosh\frac{r}{2}\Big)^{m-1}} \le C, \quad \forall r>0, \forall (2n,m). \tag{6.1}$$

The heat kernel on $\mathbb{R}^+ \times \mathrm{H}(2n,m)$, $K^{(2n,m)}(t,g,\xi)$, is a function of $(t, d(g,\xi))$, and we define $K^{(2n,m)}(t,r)$ ($t>0$, $r\ge 0$) as

$$K^{(2n,m)}(t,\varsigma) = K^{(2n,m)}(t,g,\xi), \qquad \text{with } \varsigma = d(g,\xi).$$

We have (cf. eg. (5.8) of [3]):

(1) For m even,

$$K^{(2n,m)}(t,r) = 2^{-2n-\frac{m}{2}-1}\pi^{-\frac{2n+m+1}{2}}t^{-\frac12}e^{-\frac{Q^2}{4}t}\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^{\frac{m}{2}}\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)^n e^{-\frac{r^2}{4t}}.$$

(2) For m odd,

$$K^{(2n,m)}(t,r) = 2^{-2n-\frac{m}{2}-1}\pi^{-\frac{2n+m+2}{2}}t^{-\frac12}e^{-\frac{Q^2}{4}t}$$
$$\times\int_r^{+\infty}\frac{\sinh s}{\sqrt{\cosh s-\cosh r}}\Big(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\Big)^{\frac{m+1}{2}}\Big(-\frac{1}{\sinh\frac{s}{2}}\frac{\partial}{\partial s}\Big)^n e^{-\frac{s^2}{4t}}\,ds.$$

The following observation will play an important role, but it seems that it does not exist in the literature:

By recurrence, we can show that for $k\ge 1$,

$$\begin{aligned}
&\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^k\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)^n e^{-\frac{r^2}{4t}}\\
&=\Big\{\frac{1}{2\cosh\frac{r}{2}}\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)\Big\}^k\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)^n e^{-\frac{r^2}{4t}}\\
&=\Big(2\cosh\frac{r}{2}\Big)^{-k}\sum_{j=1}^k C(k,j)\Big(2\cosh\frac{r}{2}\Big)^{j-k}\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)^{n+j}e^{-\frac{r^2}{4t}},
\end{aligned}$$

with

$$C(k,k)=1, k\ge 1, \quad C(k,1) = (2k-3)!!, k\ge 2,$$
$$C(k+1,j) = (2k-j)C(k,j) + C(k,j-1) > 0, 2\le j\le k.$$

By (4.6), we have then

$$\Big(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^k\Big(-\frac{1}{\sinh\frac{r}{2}}\frac{\partial}{\partial r}\Big)^n e^{-\frac{r^2}{4t}} \geq \sqrt{\pi t}\Big(2\cosh\frac{r}{2}\Big)^{-k}\pi^{n+k}e^{\frac{(n+k)^2}{4}t}K_{2(n+k)+1}(\frac{t}{4},\frac{r}{2}). \tag{6.2}$$

This allows us to obtain easily a lower estimate of

$$\Big[-\rho^2_{\mathbb{R}^+\times\mathrm{H}(2n,m)}+\frac{\alpha^2}{4}(n+\frac{m}{2})^2-\Delta_{\mathbb{R}^+\times\mathrm{H}(2n,m)}\Big]^{-1}, \quad \text{for } m \text{ even},$$

and of

$$\Big[-\rho^2_{\mathbb{R}^+\times\mathrm{H}(2n,m)}+\frac{\alpha^2}{4}(n+\frac{m+1}{2})^2-\Delta_{\mathbb{R}^+\times\mathrm{H}(2n,m)}\Big]^{-1}, \quad \text{for } m \text{ odd},$$

using the same argument as in the proof of Lemma 5.1.

Let $S_{\mathrm{H}(2n,m)}$ be the spherical maximal function on $\mathrm{H}(2n,m)$, which is defined for continuous function ψ and $(x,\varrho)\in\mathrm{H}(2n,m)$, by

$$S_{\mathrm{H}(2n,m)}\psi(x,\varrho)=\sup_{r>0}\frac{1}{\sigma(S^{2n-1})}\int_{\theta\in S^{2n-1}}|\psi|\Big((x,\varrho)\cdot\delta_r(\theta,0)\Big)\,d\sigma(\theta).$$

Let $1<p<2$. We have for $n+m$ big enough, for all continuous functions f, and all $g=(a,(x,\varrho))\in\mathbb{R}^+\times\mathrm{H}(2n,m)$,

$$\begin{aligned} Mf(g) &\leq c(n+m)^2\Big\{-\rho^2_{\mathbb{R}^+\times\mathrm{H}(2n,m)}+\frac{1}{4p}\Big(n+[\frac{m+1}{2}]\Big)^2-\Delta_{\mathbb{R}^+\times\mathrm{H}(2n,m)}\Big\}^{-1}(|f|)(g)\\ &\quad + c\sup_{s>0}e^{sL_Q}\Big\{M_{\mathbb{R}^m}\Big[S_{\mathrm{H}(2n,m)}f(*,(x,\cdot))\Big](\varrho)\Big\}(a), \end{aligned}$$

where the constant $c>0$ is independent of $(p,(2n,m),f,g)$.

It is quite plausible that, by a result of [38] and modifying the proof of Lemma 5 of [54], one can show that

$$\Big\|S_{\mathrm{H}(2n,m)}\Big\|_{L^p(\mathrm{H}(2n,m))\longrightarrow L^p(\mathrm{H}(2n,m))}\leq C(m,p), \quad \forall n\geq n_*(m,p). \tag{6.3}$$

It would be very interesting to know whether we have a stronger estimate as follows:

$$\Big\|S_{\mathrm{H}(2n,m)}\Big\|_{L^p(\mathrm{H}(2n,m))\longrightarrow L^p(\mathrm{H}(2n,m))}\leq C(p), \quad \forall n+m\geq l(p). \tag{6.4}$$

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7 Appendix

It would be interesting to compare the method of this paper with that in [12]. In order to obtain the L^p-boundedness for the centered Hardy-Littlewood maximal function in the setting of noncompact symmetric spaces, Clerc and Stein used the Herz majorizing principle for the part at infinity, the classic Vitali covering lemma and the Marcinkiewicz interpolation theorem for the local part. It is obvious that the Herz majorizing principle can't give any bounds for the L^p norm, and that the classic Vitali covering lemma and the Marcinkiewicz interpolation theorem do not imply the desired norm estimates as we have seen in the setting of $\mathbb{R}^n$. A natural idea to treat the (micro) local part is to compare it with the counterpart in the setting of $\mathbb{R}^n$, by a simple calculation, but we find that it is not enough for proving (1.7) in the setting of $\mathbb{H}^n$.

We note that in the setting of harmonic AN groups, $\mathbb{R}^+ \times \mathrm{H}(2n,m)$, by the result in [4], it is easy to show L^p $(1 < p < +\infty)$-dimension free estimates for the Riesz transform $\nabla(-\Delta)^{-\frac{1}{2}}$, i.e.

$$\|\nabla(-\Delta)^{-\frac{1}{2}}\|_{L^p(\mathbb{R}^+\times\mathrm{H}(2n,m))\longrightarrow L^p(\mathbb{R}^+\times\mathrm{H}(2n,m))} \leq C(p), \qquad \forall \mathbb{R}^+ \times \mathrm{H}(2n,m).$$

It is quite plausible that we have for $p > 1$

$$\|M\|_{L^p(\mathbb{R}^+\times\mathrm{H}(2n,m))\longrightarrow L^p(\mathbb{R}^+\times\mathrm{H}(2n,m))} \leq C(p), \qquad \forall \mathbb{R}^+ \times \mathrm{H}(2n,m).$$

One possible approach is to prove (6.4). Another possible approach is to obtain the first L^p $(1 < p < +\infty)$-dimension free estimates for the centered Hardy-Littlewood maximal function in the setting of S^n (the unit sphere of dimension n), M_{S^n}, and using the method of this paper. Recall that an estimate of type $\|M_{S^n}\|_{L^1\longrightarrow L^{1,\infty}} = O(n)$ has been obtained in [26] and [32].

This idea and method in [31], [33] and in this paper, can be applied to the case of some product manifolds, weighted manifolds or operators. For example, we consider the weighted manifold $\mathbb{R}^{(n,v)} = (\mathbb{R}^n, g_{\mathbb{R}^n}, e^{2\langle v,x\rangle}\, dx)$ with

$$\begin{aligned} dx \qquad & \text{the Lebesgue measure,} \\ v = (v_1, \cdots, v_n) \qquad & \text{a constant vector form } \mathbb{R}^n \setminus \{o\}, \\ g_{\mathbb{R}^n} = \sum_{i=1}^n (dx_i)^2 \qquad & \text{the Euclidean metric,} \end{aligned}$$

which may be considered as the direct product of weighted manifolds $(\mathbb{R}, g_{\mathbb{R}}, e^{2v_i t}\, dt)$ $(1 \leq i \leq n)$. The Laplace operator on $\mathbb{R}^{(n,v)}$ is the canonical Laplacian on $\mathbb{R}^n$ with drift

$$\Delta_{\mathbb{R}^{(n,v)}} = \sum_{i=1}^n \Big(\frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \Big).$$

The induced geodesic distance is the canonical Euclidean distance. It is obvious that $\mathbb{R}^{(n,v)}$ $(v \in \mathbb{R}^n \setminus \{o\})$ has the property of exponential volume growth. Let $M_{\mathbb{R}^{(n,v)}}$ denote

its centered Hardy-Littlewood maximal function. However, we have

$$\|M_{\mathbb{R}(n,v)}\|_{L^1 \longrightarrow L^{1,\infty}} \leq A(n+1)\ln(n+1), \qquad \forall v \in \mathbb{R}^n \setminus \{o\}, \forall n \geq 1,$$
$$\|M_{\mathbb{R}(n,v)}\|_{L^p \longrightarrow L^p} \leq C_p \ (1 < p \leq +\infty), \qquad \forall v \in \mathbb{R}^n \setminus \{o\}, \forall n \geq 1.$$

Recall that in the general setting of weighted manifolds, there exist some results concerning the dimension dependency of L^p bounds and even of L^p boundedness, see for example [27], [2], [14] and references therein.

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